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STABILITY OF STATIONARY SPATIALLY PERIODIC CONVECTIVE MOTIONS IN A PLANE VERTICAL LAYER

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A wide range of studies has been dedicated to the stability of plane-parallel convective motions in viscous liquid layers (see [1, 2]). It is known that in those cases where instability is of a monotonic character, it leads to development of stationary spatially periodic motions. Clever et al. [3, 4] studied stability of finite amplitude secondary motions. In [5-8] the stability of convective swell was considered, while [9] treated hexagonal cells which develop in horizontal layers due to an equilibrium crisis. In these studies stability was determined by solution of the spectral problem obtained by applying the Halerkin method to the linearized problem for perturbations. The present study will examine the stability of stationary spatially periodic motions in a planar vertical layer in the presence of lateral heating. The increments of the least stable perturbation will be determined from the time asymptote of the solution of the linearized perturbation problem, which will be constructed by the grid method [10, 11]. Calculations are performed for Prandtl number by the grid method [10, 11]. Calculations are performed for Prandtl number Pr = 1 over the Grashof number range 500 < Gr < 2000. The dependence of the increment on quasiwave number is obtained, the boundaries of the stability region are defined for spatially periodic secondary motions, and the main types of perturbations producing instability are determined.

1. We will consider an infinite vertical layer filled by a viscous incompressible fluid. On the solid boundaries of the layer ($y = \pm d$) constant but different temperatures $T = \pm\Theta$ are maintained (the x axis is directed vertically upward, and the y axis is horizontal). In dimensionless form we write the system of equations for two-dimensional convection:

$$\partial\Phi/\partial t = \Delta\Phi + \mathrm{Gr}\,\partial T/\partial y + D(\Phi, \Psi)/D(x, y); \tag{1.1}$$

$$\Delta\Psi = -\Phi; \tag{1.2}$$

$$\partial T/\partial t = (1/\mathrm{Pr})\Delta T + D(T, \Psi)/D(x, y), \tag{1.3}$$

where $D(f, g)/D(x, y) = (\partial f/\partial x)\partial g/\partial y - (\partial f/\partial y)\partial g/\partial x$; Ψ, flow function; Φ, vorticity. The similarity parameters are the Grashof number Gr and Prandtl number Pr. Assuming the flow to be closed (no pumping of liquid along the layer) the boundary conditions have the form

$$\Psi = \partial\Psi/\partial y = 0,\ T = \pm 1 \ \text{ at } \ y = \pm 1. \tag{1.4}$$

We also require that all functions remain finite at infinity

$$|\Phi|,\ |\Psi|,\ |T| < \infty \ \text{ as } \ x \to \pm\infty. \tag{1.5}$$

Boundary problem (1.1)-(1.5) always has a solution

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$$\Psi_0 = (\mathrm{Gr}/24)(1 - y^2)^2,\ T_0 = y,\ \Phi_0 = (\mathrm{Gr}/6)y(1 - y^2), \tag{1.6}$$

describing plane-parallel flow. If the Prandtl number Pr < 12, this flow loses stability with respect to monotonically increasing perturbations at $\mathrm{Gr} > \mathrm{Gr}_c \approx 500$ [1]. The nonlinear development of perturbations in the supercritical region leads to establishment of stationary spatially periodic motions [12, 13]. In [14] the stability of secondary flows was studied in the threshold region $\mathrm{Gr} \approx \mathrm{Gr}_c$; it was established that the least stable are those perturbations containing Fourier components with wave numbers close to the wave number of the main flow (Eckhaus instability). In [15, 16] the stability of secondary flows was studied for finite supercriticality $\mathrm{Gr} - \mathrm{Gr}_c$, but only with respect to perturbations with integral wave numbers. It was shown that with increase in period nonstationary motions of the standing-wave type develop.

The present study will examine stability of spatially periodic stationary motions at a finite excess above the threshold Grashof number with respect to planar perturbations of general form, not necessarily periodic.

2. Let $F_1 = (\Psi_1, \Phi_1, T_1)$ be the stationary solution of Eqs. (1.1)-(1.5) with period $2\pi/k$. For a small normal perturbation $Fe^{\lambda t} = (\Psi, \Phi, T)e^{\lambda t}$, imposed on this solution, we obtain the linear boundary problem

$$\lambda\Phi = \Delta\Phi + \mathrm{Gr}\frac{\partial T}{\partial y} + \frac{D(\Phi, \Psi_1)}{D(x, y)} + \frac{D(\Phi_1, \Psi)}{D(x, y)}, \tag{2.1}$$

$$\Delta\Psi = -\Phi,\quad \lambda T = \mathrm{Pr}^{-1}\Delta T + \frac{D(T, \Psi_1)}{D(x, y)} + \frac{D(T_1, \Psi)}{D(x, y)};$$

$$\Psi = \partial\Psi/\partial y = T = 0 \ \text{at} \ \ y = \pm 1,\ |\Phi|,\ |\Psi|,\ |T| < \infty \ \text{at} \ \ x = \pm\infty. \tag{2.2}$$

In the case of a plane-parallel main motion $F_0(y)$, independent of the value of the longitudinal coordinate x, small perturbations can be represented in the form $F(x, y) = f(y)e^{ikx}$ and can be described by Orr–Sommerfeld-type ordinary differential equations. For spatially periodic motion perturbations such as a separation of variables does not occur, since in Eq. (2.1) functions periodic in x with period $2\pi/k$ appear as coefficients. Nevertheless, the form of the solution remains quite specific and follows from the group properties of the boundary problem (2.1), (2.2) [17]:

$$F(x, y) = f(x, y)e^{iqx},\ f = (\psi, \varphi, \theta),\ f(x + 2\pi/k, y) = f(x, y) \tag{2.3}$$

(Flocke–Bloch function). The real parameter q (which we will term the quasiwave number of the perturbation) is defined to the accuracy of an integer k, and can be chosen within the range $|q| \leqslant k/2$.

Substituting Eq. (2.3) in Eqs. (2.1), (2.2) we obtain a boundary problem for the eigenfunctions, containing q as an independent parameter:

$$\lambda\varphi = \Delta\varphi + 2iq\frac{\partial\varphi}{\partial x} - q^2\varphi + \mathrm{Gr}\frac{\partial\theta}{\partial y} + \frac{D(\varphi, \Psi_1)}{D(x, y)} + \frac{D(\Phi_1, \psi)}{D(x, y)} + iq\frac{\partial\Psi_1}{\partial y}\varphi - iq\frac{\partial\Phi_1}{\partial y}\psi; \tag{2.4}$$

$$\Delta\psi + 2iq\frac{\partial\psi}{\partial x} - q^2\psi + \varphi = 0; \tag{2.5}$$

$$\lambda\theta = \mathrm{Pr}^{-1}\left(\Delta\theta + 2iq\frac{\partial\theta}{\partial x} - q^2\theta\right) + \frac{D(\theta, \Psi_1)}{D(x, y)} + \frac{D(T_1, \psi)}{D(x, y)} + iq\frac{\partial\Psi_1}{\partial y}\theta - iq\frac{\partial T_1}{\partial y}\psi; \tag{2.6}$$

$$\psi = \partial\psi/\partial y = \theta = 0,\quad \psi(x + 2\pi/k, y) = \psi(x, y),\ \varphi(x + 2\pi/k, y) = \varphi(x, y),\ \theta(x + 2\pi/k, y) = \theta(x, y) \ \text{at} \ \ y = \pm 1. \tag{2.7}$$

Let $f_n(x, y)$, n = 1, 2, ... be eigenfunctions of this problem, renormalized such that for all corresponding increments $\mathrm{Re}\,\lambda_n \geqslant \mathrm{Re}\,\lambda_{n+1}$. The main motion with period $2\pi/k$ is unstable if for any q $\mathrm{Re}\,\lambda_1 > 0$.*

We will now consider certain symmetry properties of the functions $F_1(x, y)$ and $f(x, y)$.

*In contrast to the Orr–Sommerfeld problem, the completeness of the eigenfunction system has not been proven for the two-dimensional problem Eqs. (2.4)-(2.7). Therefore, strictly speaking, the instability condition formulated is only sufficient. For the future, however, we will assume (as is usually done) that the set of functions f_n is complete.

We introduce the transform

$$\Pi(\psi(x, y), \varphi(x, y), \theta(x, y)) \equiv (\bar{\psi}(-x, -y), \bar{\varphi}(-x, -y), -\bar{\theta}(-x, -y)).$$

From Eqs. (1.1)-(1.4) it is evident that the functions Ψ_1 and Φ_1 can be chosen symmetric, and T_1 can be chosen antisymmetric with respect to inversion, i.e.,

$$\Pi(F_1) = F_1. \tag{2.8}$$

Further, it follows from Eqs. (2.4)-(2.7) that if f_1 is the eigenfunction corresponding to the increment λ_1, then the function $\Pi(f_1)$ is an eigenfunction and corresponds to the increment $\bar{\lambda}_1$. In the case of a real nondegenerate value of λ_1 we may take $f_1 = \Pi(f_1)$, i.e.,

$$\varphi_1(x, y) = \bar{\varphi}_1(-x, -y), \ \psi_1(x, y) = \bar{\psi}_1(-x, -y),$$
$$\theta_1(x, y) = -\bar{\theta}_1(-x, -y).$$

3. Calculation of stationary secondary motions can be performed numerically using the grid method with Eqs. (1.1)-(1.5). This same method can be used to construct small perturbations.

We will consider the nonstationary problem obtained by replacing λ by the operator $\partial/\partial t$ in Eqs. (2.4)-(2.7), supplemented by a special initial condition. If

$$f(x, y, 0) = \sum_{n=1}^{\infty} a_n f_n(x, y),$$

where a_n are constant coefficients, then the solution of the Cauchy problem has the form

$$f(x, y, t) = \sum_{n=1}^{\infty} a_n f_n(x, y) e^{\lambda_n t}.$$

We note that if the function f(x, y, 0) has the property that

$$\Pi(f(x, y, 0)) = f(x, y, 0),$$

then this symmetry property is maintained at following moments in time

$$\Pi(f(x, y, t)) = f(x, y, t). \tag{3.1}$$

The increment of the least stable mode is determined by examining the time asymptote of the given solution. If the value of λ_1 is real, then in the absence of random degeneration ($\lambda_1 = \lambda_2$) as $t \to \infty$

$$f(x, y, t) \sim a_1 f_1(x, y) e^{\lambda_1 t}.$$

Then the value of $\lambda_1(q)$ can be calculated, for example, as

$$\lambda_1(q) = \lim_{t\to\infty} \frac{\partial}{\partial t} \ln|\psi_r(x_0, y_0, t)|, \tag{3.2}$$

where $\psi_r \equiv \mathrm{Re}\,\psi$; the choice of points x_0, y_0 is arbitrary. In the case of a complex increment λ_1 as $t \to \infty$

$$f(x, y, t) \sim a_1 f_1 e^{\lambda_1 t} + a_2 f_2 e^{\bar{\lambda}_1 t}, \tag{3.3}$$

where $f_2 = \Pi(f_1)$. It can be seen that at $a_2 = \bar{a}_1$ the function f(x, y, t) has the property of Eq. (3.1). Introducing the notation $f_+ = f_1 + f_2$, $f_- = f_1 - f_2$ and setting $a_1 = a \exp(i\alpha)$, we write the solution (3.3) in the form

$$f(x, y, t) \sim a e^{\mathrm{Re}\lambda_1 \cdot t} [f_+ \cos(\mathrm{Im}\,\lambda_1 \cdot t + \alpha) + i f_- \sin(\mathrm{Im}\,\lambda_1 \cdot t + \alpha)].$$

To calculate the increment in this case we may use the expressions:

$$\mathrm{Re}\,\lambda_1(q) = \lim_{t\to\infty} \frac{1}{2} \left[\frac{\dddot{\psi}_r \psi_r - \ddot{\psi}_r \dot{\psi}_r}{\ddot{\psi}_r \psi_r - \dot{\psi}_r^2} \right]_{x=x_0,\, y=y_0},$$

$$\mathrm{Im}\,\lambda_1(q) = \lim_{t\to\infty} \left[\frac{1}{\psi_r} (2\dot{\psi}_r \mathrm{Re}\,\lambda_1 - \ddot{\psi}_r) - (\mathrm{Re}\,\lambda_1)^2 \right]^{1/2}_{x=x_0,\, y=y_0}$$

(the dot denotes differentiation with respect to t). We stress that the increment can be determined from the asymptote of the solutions satisfying the additional symmetry property, Eq. (3.1).

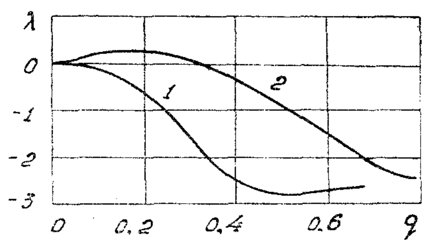

Fig. 1.

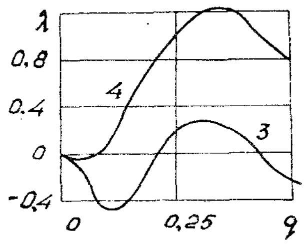

Fig. 2.

The equations describing the evolution of vorticity and temperature for the main flow and for perturbations having identical structure are:

$$\partial u/\partial t = a(\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2) + b_1 \partial u/\partial x + b_2 \partial u/\partial y + cu + d$$

and can be solved by the partial step method with longitudinal–transverse drive [18]. The Poisson equation is solved by the Liebman iteration method with successive upper relaxation. In view of their identical periodicity the functions F_1 and f are constructed on one grid pattern. Solutions are sought which satisfy conditions (2.8), (3.1); the problem is solved in half of the layer $-1 \leqslant y \leqslant 0$; at y = 0 we impose the inverse symmetry (or antisymmetry) condition for the corresponding functions.

Due to the exponential growth of perturbations, the values of the functions f^n and f^{n+1}, taken at one and the same points in neighboring time layers will differ even in the steady-state regime. In connection with this, special measures were taken to eliminate factors disrupting the implicitness of the method and degrading the approximation.

In particular, to perform the vertical drive cyclical boundary conditions [19] were employed. In approximating the vorticity in the case of real λ the following procedure was used:

$$\varphi^{n+1}|_{y=-1} = -(2/h^2)\psi^n|_{y=-1+h}e^{\lambda\tau}, \tag{3.4}$$

where n is the number of the time layer; h is the spatial grid step along the y axis, and τ is the time step. Equation (3.4) is in fact Tom's formula, modified by consideration of the asymptote of the solution at large times. The value of λ is determined from Eq. (3.2). In the case of complex λ another procedure is used:

$$\varphi^{n+1}|_{y=-1} = -(2/h^2)(2\psi^n - \psi^{n-1})_{y=-1+h}.$$

A more detailed description of this calculation method is given in [11].

4. We will turn to an evaluation of the results of calculations performed at fixed Prandtl number (Pr = 1). Figures 1-3 show characteristic dependences of the increment of the least stable mode on quasiwave number $|q| < k/2$. Because of the even nature of the increments $\lambda(q) = \lambda(-q)$ the graphs show only the half of the region with q > 0 (this property can be shown most easily by using the notation f = Re f + i Im f in Eqs. (2.4)-(2.6) and writing separate equations for real and imaginary components). Figure 4 shows a composite graph of stability of spatially periodic motions in the plane of the parameters Gr, k.

The λ(q) curves of Fig. 1 were constructed for Gr = 550. Curve 1 corresponds to stable

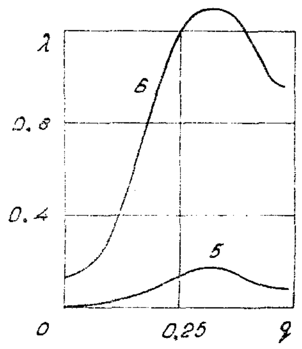

Fig. 3.

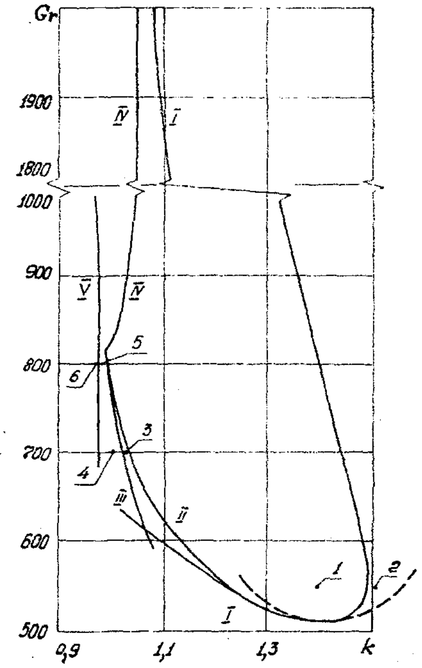

Fig. 4.

motion (k = 1.4), while curve 2 is unstable (k = 1.57); in Fig. 4 these motions are shown by points 1 and 2, respectively. As is evident from the upper graph, the least stable perturbations are those with small q (in other words, with wave numbers close to the wave number of the main flow), Eckhaus type perturbations. In Fig. 4 the boundary dividing the regions of stability and instability with respect to perturbations of this type is denoted by the symbol I; it is defined by the condition $\partial^2\lambda/\partial q^2 = 0$. The dashed line shows the same curve, as obtained analytically in [14]. It is obvious that the region of applicability of analytical methods in the given problem is quite limited — even at slight supercriticality both branches of the curve I deviate markedly to the left of the dashed line.

With growth in Grashof number the right-hand (short wave) boundary of the stability region proves always to be related to Eckhaus type perturbations. As for the left-hand boundary (long-wave region), there is greater variation in the type of instability. Figure 2 shows spectra of motions with wave numbers k = 1.03 (curve 3) and k = 1 (curve 4) for Gr = 700. In both cases the least stable perturbations are those with finite q. The stability boundary with respect to perturbations of that type is shown by line II in Fig. 4.

With increase in Grashof number the value of q corresponding to maximum λ shifts to the right until it coincides (at Gr ≈ 800) with the boundary of the zone q = k/2. The stability curve for perturbations with q = k/2 is denoted by the symbol III.

With further increase in Gr the perturbations with q = k/2 remain the least stable, and become oscillatory. The stability curve III breaks off at the point where Im $\lambda \neq 0$ and continues further as line IV (Fig. 4). We note that at finite q < k/2 (less unstable) the complex branch λ(q) can split again into two real curves.

Further increase in Gr does not lead to qualitatively new types of instability. The boundaries I and IV slowly move toward each other, while curve I approaches an almost vertical asymptote. The region of stable secondary motions gradually narrows, and at Gr ≈ 2000 there is only a very narrow interval of wave numbers in the vicinity of k ≈ 1.05, in which spatially periodic motions still remain stable. We stress that the stability region still does exist at high supercriticality in the form of a very narrow band. In practice, a motion with a fully defined wave number remains.

In conclusion, we will touch briefly on yet another type of perturbation observed in the problem under consideration, which, however, does not prove to be the least stable in any parameter range. The stability boundary with respect to perturbations of this type is given in Fig. 4 by curve V. Upon passage through this curve perturbations with q = 0 begin to increase monotonically, i.e., those perturbations with the same period as the main flow. The function $\lambda(q)$ for motion from this region is shown in Fig. 3 (curve 6, Gr = 800, k = 0.98). We note that direct numerical solution of system (1.1)-(1.4) with periodic boundary conditions in x for parameter values taken from the region of the lefthand boundary V show instability of inverse symmetric stationary main motion, leading to development of flows in the form of periodic vortex systems moving upward or downward in the layer, and not having inverse symmetry.

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